Comparison results for a nonlocal singular elliptic problem

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Joint work with B. Brandolini, V. Ferone, B. Volzone

January 30, 2023

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where:

- Ω is a bounded, open set in $\mathbb{R}^N$, $N > 2s$, $0 < s < 1$;
- the nonlocal operator in the left hand side, that is $(-\Delta)^s$, is the fractional Laplacian operator defined, up to a normalization factor, by the Riesz potential as

$$-(-\Delta)^s u(x) := P.V. \int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{y^{N+2s}} dy, \quad x \in \mathbb{R}^N;$$

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B.Brandolini, I. de Bonis, V. Ferone, B. Volzone, *Comparison results for a nonlocal singular elliptic problem*, submitted.

Aim

Our aim is to use symmetrization techniques in order to get a comparison result between the weak solution to problem (1) and the weak solution v to a symmetrized problem, defined in the ball $\Omega^\star$ centered at the origin having the same measure as Ω, which stays in the same class as the original one (namely singular and nonlocal).

By Talenti's seminal paper it is well known that, if $u \in H_0^1(\Omega)$ and $v \in H_0^1(\Omega^\star)$ solve

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad \text{and} \quad \begin{cases} -\Delta v = f^\star & \text{in } \Omega^\star \\ v = 0 & \text{on } \partial\Omega^\star, \end{cases}$$

respectively, then

$$u^\star(x) \leq v(x), \qquad x \in \Omega^\star \tag{2}$$

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From the previous estimate derive, for istance, that
any Lebesgue norm of u is bounded from above by the same Lebesgue norm of v.
This means that the fact to estimate the solution u of a Dirichlet problem in Ω is solved once we can estimate the solution v of a symmetrized problem, which is much easier to handle with, since it is a one-dimensional problem.

G. Talenti, *Elliptic equations and rearrangements*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 3 (1976), 697–718.

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Local context: Symmetrization techniques have been applied to local, singular problems like (1) when, on the left-hand side, the Laplacian operator replaces the fractional one (see [BCT]).

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B. Brandolini, F. Chiacchio, C. Trombetti, *Symmetrization for singular semilinear elliptic equations,* Annali di Matematica 193 (2014) 389–404.

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B. Brandolini, F. Chiacchio, C. Trombetti, *Symmetrization for singular semilinear elliptic equations,* Annali di Matematica 193 (2014) 389–404.

G. di Blasio, and B. Volzone, *Comparison and regularity results for the fractional Laplacian via Symmetrization methods*, Journal of Differential Equations 253 (2012), 2593–2615.

Our approach

We adopt the direct approach introduced in Volzone and Ferone's paper (2021) where the problem (1) is considered in the case $\gamma = 0$.

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This direct approach
does not use the local interpretation of the fractional Laplacian described above, while it makes a clever use of a nonlocal version of the classical Pólya-Szegő inequality, plus a sophisticated representation of the fractional Laplacian of a spherical mean function in $(N+2)$ dimensions, which allows to conclude by a maximum principle argument.

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The novelty in our paper is in the fact that the above mentioned interpretation is avoided in the proofs of our new results, thus in this sense they offer an alternative to the crucial part of the proof of the main theorem of [FV] (comparison result).

V. Ferone, and B. Volzone, *Symmetrization for fractional elliptic problems: a direct approach*, Arch. Rational Mech. Anal. 239 (2021), 1733–1770.

Main results

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Theorem ((1) B. Brandolini, I. de Bonis, V. Ferone, B. Volzone, 2022)

Let $s \in (0,1)$, $N \geq 2$, $\gamma > 0$ and assume that $f \in L^\infty(\Omega)$, $f \geq 0$. If u is a weak solution to problem (1) *and v is the solution to the symmetrized problem*

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$$\begin{cases} (-\Delta)^s v = \dfrac{||f||_{L^\infty(\Omega^\star)}}{v^\gamma} & \text{in } \Omega^\star \\ v > 0 & \text{in } \Omega^\star \\ v = 0 & \text{on } \mathbb{R}^N \setminus \Omega^\star, \end{cases} \tag{3}$$

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then

$$\int_{B_r(0)} u^\star(x)\,\mathrm{d}x \leq \int_{B_r(0)} v^\star(x)\,\mathrm{d}x, \qquad r > 0. \tag{4}$$

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Theorem ((2) B. Brandolini, I. de Bonis, V. Ferone, B. Volzone, 2022)

Let $s \in (0,1)$, $N \geq 2$, $\gamma > 0$ and assume that $f \in L^1(\Omega)$, $f \geq 0$. If u is a weak solution to problem (1) and v is the solution to the following problem

$$\begin{cases} (-\Delta)^s v = (\gamma+1) f^\star & \text{in } \Omega^\star \\ v > 0 & \text{in } \Omega^\star \\ v = 0 & \text{on } \mathbb{R}^N \setminus \Omega^\star, \end{cases} \tag{5}$$

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then

$$\int_{B_r(0)} u^\star(x)^{\gamma+1}\, \mathrm{d}x \leq \int_{B_r(0)} v^\star(x)\, \mathrm{d}x, \qquad r > 0. \tag{6}$$

Functional setting and preliminaries

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Let $s \in (0,1)$. For any open set Ω and any measurable function u on Ω, we introduce the fractional Gagliardo seminorm

$$[u]_{H^s(\Omega)} = \left(\iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} \,\mathrm{d}x\,\mathrm{d}y \right)^{1/2}.$$

Then we define the fractional Sobolev space $H^s(\Omega)$ as the space

$$H^s(\Omega) = \left\{ u \in L^2(\Omega) : [u]_{H^s(\Omega)} < \infty \right\},$$

endowed with the norm

$$\|u\|_{H^s(\Omega)} = \|u\|_{L^2(\Omega)} + [u]_{H^s(\Omega)}.$$

We denote by $H^s_0(\Omega)$ the closure of $C_c^\infty(\Omega)$ in the $H^s(\Omega)$ topology. Moreover, we will define the space

$$H^s_{\mathrm{loc}}(\Omega) = \left\{ u : \Omega \to \mathbb{R} : u_{|K} \in H^s(K), \text{ for all } K \subset\subset \Omega \right\}.$$

There is a strict connection between the space $H^s(\mathbb{R}^N)$ and the fractional Laplacian operator $(-\Delta)^s$. For any $s \in (0,1)$ and $u \in \mathscr{S}$ (the classical Schwartz class), the fractional Laplacian operator is defined as

$$(-\Delta)^s u = \gamma(N,s)\,\mathrm{P.V.}\int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\,\mathrm{d}y,$$

where

$$\gamma(N,s) = \frac{s\,2^{2s}\Gamma\left(\frac{N+2s}{2}\right)}{\pi^{\frac{N}{2}}\Gamma(1-s)}. \tag{7}$$

We are interested to Dirichlet problems defined in bounded domains. To this aim, we consider the space $X_0^s(\Omega)$, defined as

$$X_0^s(\Omega) = \left\{ u \in H^s(\mathbb{R}^N) : \; u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \right\},$$

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endowed with the Gagliardo seminorm

$$||u||_{X_0^s(\Omega)} = [u]_{H^s(\mathbb{R}^N)} = \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, \mathrm{d}x \, \mathrm{d}y \right)^{1/2}.$$

From the definition of $X_0^s(\Omega)$ it easily follows that for each $u \in X_0^s(\Omega)$

$$||u||_{X_0^s(\Omega)} = \left(\iint_Q \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, \mathrm{d}x \, \mathrm{d}y \right)^{\frac{1}{2}}$$

where $Q = \mathbb{R}^{2N} \setminus (\mathscr{C}\Omega \times \mathscr{C}\Omega)$ and $\mathscr{C}\Omega = \mathbb{R}^N \setminus \Omega$.

Some notions about Schwarz symmetrization

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$$\mu_u(t) := |\{x \in \Omega : |u(x)| > t\}| < +\infty, \quad \text{for every } t > 0,$$

we define the radially symmetric, decreasing rearrangement of u, also known as the Schwarz decreasing rearrangement of u, as

$$u^\star(x) = \sup\{t \geq 0 : \mu_u(t) > \omega_N |x|^N\}, \qquad x \in \Omega^\star,$$

where ω_N is the measure of the unitary ball in $\mathbb{R}^N$, and $\Omega^\star$ is the ball (centered at the origin) having the same measure as Ω.

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Let $u, v \in L^1_{loc}(\mathbb{R}^N)$. We say that u is *less concentrated* than v, and we write $u \prec v$, if for every $r > 0$ we have

$$\int_{B_r(0)} u^\star(x)\,\mathrm{d}x \leq \int_{B_r(0)} v^\star(x)\,\mathrm{d}x.$$

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Clearly, this definition can be adapted to functions defined in an open subset Ω of $\mathbb{R}^N$, by extending the functions to zero outside Ω.

The partial order relationship $\prec$ is called *comparison of mass concentrations* and it satisfies some nice properties.

A. Alvino, P. L. Lions, and G. Trombetti, *On optimization problems with prescribed rearrangements*, Nonlinear Anal. Theory Methods Appl. 13 (1989), 185–220.

Definition of solution

Before proving our main result we need to specify the notion of solution to problem (1). Note that, due to the lack of regularity of solutions near the boundary, the notion of solution has to be understood in the weak distributional meaning, for test functions compactly supported in the domain. Furthermore, the nonlocal nature of the operator has to be taken into account.
We will adopt the following notion of solution contained in

A. Canino, L. Montoro, B. Sciunzi and M. Squassina, *Nonlocal problems with singular nonlinearity*, Bull. Sci. Math. 141 (2017), 223–250.

Definition

We say that a positive function $u \in H^s_{\mathrm{loc}}(\Omega) \cap L^1(\Omega)$ is a weak solution to problem (1) if

$$u^{\max\{\frac{\gamma+1}{2},1\}} \in X^s_0(\Omega), \qquad \frac{f}{u^\gamma} \in L^1_{\mathrm{loc}}(\Omega),$$

and, for every nonnegative $\varphi \in C^\infty_c(\Omega)$, we have

$$\frac{\gamma(N,s)}{2} \iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}} \,\mathrm{d}x\,\mathrm{d}y = \int_\Omega \frac{f(x)}{u(x)^\gamma}\varphi(x)\,\mathrm{d}x,$$

with $\gamma(N,s)$ defined in (7).

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the authors prove the existence of a weak solution to problem (1) with $\operatorname{ess\,inf}_K u > 0$ for every compact set $K \subset\subset \Omega$, distinguishing two cases according to the value of γ:

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1) (mildly singular) when $0 < \gamma \leq 1$ and $f \in L^p(\Omega)$, then there exists a solution $u \in X_0^s(\Omega)$;

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1) (mildly singular) when $0 < \gamma \leq 1$ and $f \in L^p(\Omega)$, then there exists a solution $u \in X_0^s(\Omega)$;

2) (strongly singular) when $\gamma > 1$ and $f \in L^1(\Omega)$, then there exists a solution $u \in H^s_{\mathrm{loc}}(\Omega) \cap L^1(\Omega)$ such that $u^{\frac{\gamma+1}{2}} \in X_0^s(\Omega)$.

In the same paper the authors also discuss the uniqueness of such solutions. Since the way of understanding the boundary condition is not unambiguous, they start with the following:

Definition

Let u be such that $u = 0$ in $\mathbb{R}^N \setminus \Omega$. We say that $u \leq 0$ on $\partial\Omega$ if, for every $\varepsilon > 0$, it follows that

$$(u - \varepsilon)_+ \in X_0^s(\Omega).$$

We will say that $u = 0$ on $\partial\Omega$ if u is nonnegative and $u \leq 0$ on $\partial\Omega$.

They prove that if $\gamma > 0$ and $f \in L^1(\Omega)$, there exists at most one weak solution to problem (1).

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Step 1. Approximating problems

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For every $k \in \mathbb{N}$ we define $f_k := \min\{f(x), k\}$ and we consider the following sequence of nonsingular approximating problems

$$\begin{cases} (-\Delta)^s(u_k) = \dfrac{f_k}{(u_k + \frac{1}{k})^\gamma} & \text{in } \Omega \\ u_k > 0 & \text{in } \Omega \\ u_k = 0 & \text{on } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{8}$$

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For every $k \in \mathbb{N}$ problem (8) has a nonnegative solution belonging to $X_0^s(\Omega) \cap L^\infty(\Omega)$, which means that

$$\frac{\gamma(N,s)}{2} \int\int_Q \frac{(u_k(x) - u_k(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} \,\mathrm{d}x\,\mathrm{d}y = \int_\Omega \frac{f_k(x)}{\left(u_k(x) + \frac{1}{k}\right)^\gamma} \,\mathrm{d}x \tag{9}$$

for every nonnegative $\varphi \in X_0^s(\Omega)$.

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Moreover, the sequence u_k is increasing, $u_k > 0$ in Ω, and, for every subset $\omega \subset\subset \Omega$, there exists a positive constant c_ω, independent of k, such that $u_k(x) \geq c_\omega > 0$ for every $x \in \omega$ and $k \in \mathbb{N}$.

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Step 2. Reduction to the radial case

We follow a 'direct approach' introduced by Ferone and Volzone (2021).
Let $0 \leq t < ||u_k||_{L^\infty(\Omega)}$ and $h > 0$. We consider the following test function

$$\varphi(x) = \mathcal{G}_{t,h}(u_k(x)),$$

where $\mathcal{G}_{t,h}(\theta)$ is defined as follows:

$$\mathcal{G}_{t,h}(\theta) = \begin{cases} h & \text{if } \theta > t + h \\ \theta - t & \text{if } t < \theta \leq t + h \\ 0 & \text{if } \theta \leq t. \end{cases}$$

V. Ferone, and B. Volzone, *Symmetrization for fractional elliptic problems: a direct approach,* Arch. Rational Mech. Anal. 239 (2021), 1733–1770.

We explicitly observe that $\mathcal{G}_{t,h}(\theta) \in X_0^s(\Omega)$, so we can use it in the weak formulation of solution, obtaining

$$\frac{\gamma(N,s)}{2} \int\int_{\mathbb{R}^{2N}} \frac{(u_k(x)-u_k(y))\,(\mathcal{G}_{t,h}(u_k(x))-\mathcal{G}_{t,h}(u_k(y)))}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_\Omega \frac{f_k(x)}{\left(u_k(x)+\frac{1}{k}\right)^\gamma}\mathcal{G}_{t,h}(u_k(x))\,\mathrm{d}x.$$

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$$= \int_\Omega \frac{f_k(x)}{\left(u_k(x) + \frac{1}{k}\right)^\gamma} \mathcal{G}_{t,h}(u_k(x))\,\mathrm{d}x.$$

We start by writing

$$\int\int_{\mathbb{R}^{2N}} \frac{(u_k(x) - u_k(y))\,(\mathcal{G}_{t,h}(u_k(x)) - \mathcal{G}_{t,h}(u_k(y)))}{|x-y|^{N+2s}} \,\mathrm{d}x\,\mathrm{d}y$$

$$= \frac{1}{\Gamma\left(\frac{N+2s}{2}\right)} \int_0^\infty \mathcal{I}_\alpha[u_k, t, h]\alpha^{\frac{N+2s}{2}-1}\,\mathrm{d}\alpha,$$

where

$$\mathcal{I}_\alpha[u_k, t, h] = \int\int_{\mathbb{R}^{2N}} (u_k(x) - u_k(y))\,(\mathcal{G}_{t,h}(u_k(x) - \mathcal{G}_{t,h}(u_k(y))\, e^{-\alpha|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y.$$

Riesz's general rearrangement inequality with the choices

$$F(u_k, v_k) = u_k^2 + v_k^2 - (u_k - v_k)\left(\mathcal{G}_{t,h}(u_k) - \mathcal{G}_{t,h}(v_k)\right), W_\alpha(x) = e^{-\alpha|x|^2}, a = 1, b = -1,$$

implies

$$\int\int_{\mathbb{R}^{2N}} F(u_k(x), u_k(y)) W_\alpha(x-y)\, \mathrm{d}x\, \mathrm{d}y \le \int\int_{\mathbb{R}^{2N}} F(u_k^\star(x), u_k^\star(y)) W_\alpha(x-y)\, \mathrm{d}x\, \mathrm{d}y,$$

which immediately gives

$$\mathcal{I}_\alpha[u_k, t, h] \ge \mathcal{I}_\alpha[u_k^\star, t, h].$$

Hence

$$\int\int_{\mathbb{R}^{2N}} \frac{(u_k(x) - u_k(y))\left(\mathcal{G}_{t,h}(u_k(x)) - \mathcal{G}_{t,h}(u_k(y))\right)}{|x-y|^{N+2s}}\, \mathrm{d}x\, \mathrm{d}y$$
$$\ge \int\int_{\mathbb{R}^{2N}} \frac{(u_k^\star(x) - u_k^\star(y))\left(\mathcal{G}_{t,h}(u_k^\star(x)) - \mathcal{G}_{t,h}(u_k^\star(y))\right)}{|x-y|^{N+2s}}\, \mathrm{d}x\, \mathrm{d}y.$$

If $\mathfrak{u}_k(x) = \mathfrak{u}_k(|x|)$ will stand for $u_k^\star(x)$, changing the variables we obtain

$$\int\int_{Q^\star} \frac{(u_k^\star(x) - u_k^\star(y))\,(\mathcal{G}_{t,h}(u_k^\star(x)) - \mathcal{G}_{t,h}(u_k^\star(y)))}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y$$

$$= N\omega_N \int_0^{+\infty} \left(\int_0^{+\infty} (\mathfrak{u}_k(r) - \mathfrak{u}_k(\rho))\,(\mathcal{G}_{t,h}(\mathfrak{u}_k(r)) - \mathcal{G}_{t,h}(\mathfrak{u}_k(\rho))) \right.$$

$$\left. \Theta_{N,s}(r,\rho)\rho^{N-1}\,\mathrm{d}\rho r^{N-1}\,\mathrm{d}r, \right.$$

where $\Theta_{N,s}(r,\rho)$ is the function given by

$$\Theta_{N,s}(r,\rho) = \frac{1}{N\omega_N}\int_{|x'|=1} \left(\int_{|y'|=1} \frac{1}{|r\,x' - \rho\,y'|^{N+2s}}\,\mathrm{d}\mathcal{H}^{N-1}(y') \right) \mathrm{d}\mathcal{H}^{N-1}(x')$$

defined for $r, \rho > 0$.

Since the previous internal integral does not depend on x', we can compute it by choosing any fixed x' and we obtain

$$\begin{aligned}\Theta_{N,s}(r,\rho) &= \int_{|y'|=1} \frac{1}{|r\,x' - \rho\,y'|^{N+2s}}\, \mathrm{d}\mathcal{H}^{N-1}(y') \\ &= \frac{2\pi^{\frac{N-1}{2}}}{\Gamma\left(\frac{N-1}{2}\right)} \int_0^\pi \frac{\sin^{N-2}\theta}{(r^2 - 2r\rho\cos\theta + \rho^2)^{\frac{N+2s}{2}}}\, \mathrm{d}\theta. \end{aligned} \tag{10}$$

Identity (10) immediately infers that $\Theta_{N,s}(r,\rho)$ is symmetric, that is

$$\Theta_{N,s}(r,\rho) = \Theta_{N,s}(\rho,r), \qquad r,\rho > 0.$$

Moreover,

$$\Theta_{N,s}(r,\rho)=\begin{cases}\dfrac{2\pi^{\frac{N-1}{2}}}{\Gamma\left(\frac{N-1}{2}\right)}\rho^{-N-2s}{}_2F_1\left(\frac{N+2s}{2},s+1;\frac{N}{2};\frac{r^2}{\rho^2}\right) & \text{if } 0\le r<\rho<+\infty\\[2ex] \dfrac{2\pi^{\frac{N-1}{2}}}{\Gamma\left(\frac{N-1}{2}\right)}r^{-N-2s}{}_2F_1\left(\frac{N+2s}{2},s+1;\frac{N}{2};\frac{\rho^2}{r^2}\right) & \text{if } 0\le \rho<r<+\infty,\end{cases} \tag{11}$$

where ${}_2F_1(a,b;c;x)$ is the hypergeometric function defined by

$${}_2F_1(a,b;c;x)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 \tau^{b-1}(1-\tau)^{c-b-1}(1-x\tau)^{-a}\,\mathrm{d}\tau,$$

$$c>b>0, 0<\tau<1.$$

It is well-known that

$${}_2F_1'(a,b;c;x)=\frac{ab}{c}{}_2F_1(a+1,b+1;c+1;x).$$

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Finally, using (11), we have the following asymptotic behaviors:

$$\Theta_{N,s}(r,\rho) \sim \frac{1}{|r-\rho|^{1+2s}} \qquad \text{as } |r-\rho| \to 0,$$

$$\Theta_{N,s}(r,\rho) \sim \frac{1}{r^{N+2s}} \quad \text{as } r \to +\infty,$$

$$\Theta_{N,s}(r,\rho) \sim \frac{1}{\rho^{N+2s}} \quad \text{as } \rho \to +\infty.$$

We split the integral in the right-hand side of last equality into the sum

$$\int_0^{+\infty}\left(\int_0^{+\infty}(\mathrm{u}_k(r)-\mathrm{u}_k(\rho))\left(\mathcal{G}_{t,h}(\mathrm{u}_k(r))-\mathcal{G}_{t,h}(\mathrm{u}_k(\rho))\right)\Theta_{N,s}(r,\rho)\rho^{N-1}\,\mathrm{d}\rho\right)r^{N-1}$$

$$=\mathcal{I}^1+2\mathcal{I}^2+2\mathcal{I}^3+2h\mathcal{I}^4,$$

where

$$\begin{aligned}
\mathcal{I}^1 &= \int_{r(t+h)}^{r(t)}\left(\int_{r(t+h)}^{r(t)}(\mathrm{u}_k(r)-\mathrm{u}_k(\rho))^2\Theta_{N,s}(r,\rho)\rho^{N-1}\,\mathrm{d}\rho\right)r^{N-1}\,\mathrm{d}r,\\
\mathcal{I}^2 &= \int_{0}^{r(t+h)}\left(\int_{r(t+h)}^{r(t)}(\mathrm{u}_k(r)-\mathrm{u}_k(\rho))(h-\mathrm{u}_k(\rho)+t)\Theta_{N,s}(r,\rho)\rho^{N-1}\,\mathrm{d}\rho\right)r^{N-1}\\
\mathcal{I}^3 &= \int_{r(t)}^{+\infty}\left(\int_{r(t+h)}^{r(t)}(\mathrm{u}_k(r)-\mathrm{u}_k(\rho))(-\mathrm{u}_k(\rho)+t)\Theta_{N,s}(r,\rho)\rho^{N-1}\,\mathrm{d}\rho\right)r^{N-1}\,\mathrm{d}r,\\
\mathcal{I}^4 &= \int_{0}^{r(t+h)}\left(\int_{r(t)}^{+\infty}(\mathrm{u}_k(r)-\mathrm{u}_k(\rho))\Theta_{N,s}(r,\rho)\rho^{N-1}\,\mathrm{d}\rho\right)r^{N-1}\,\mathrm{d}r,
\end{aligned}$$

with $\mathrm{u}_k(r(t))=t$ and $\mathrm{u}_k(r(t+h))=t+h$.

It can be proved that

$$\frac{\mathcal{I}^1}{h} \to 0, \quad \text{as } h \to 0^+,$$
$$\frac{\mathcal{I}^2}{h} \to 0, \quad \text{as } h \to 0^+,$$
$$\frac{\mathcal{I}^3}{h} \to 0, \quad \text{as } h \to 0^+.$$

It follows that

$$\lim_{h\to 0^+} \frac{1}{h} \iint_{\mathbb{R}^{2N}} \frac{[u_k^\star(x) - u_k^\star(y)]\,[\mathcal{G}_{t,h}(u_k^\star(x)) - \mathcal{G}_{t,h}(u_k^\star(y))]}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y$$
$$= 2N\omega_N \int_0^{r(t)} \left(\int_{r(t)}^{+\infty} (\mathfrak{u}_k(r) - \mathfrak{u}_k(\rho))\Theta_{N,s}(r,\rho)\rho^{N-1}\,\mathrm{d}\rho \right) r^{N-1}\,\mathrm{d}r.$$

We now focus on the singular lower order term. Since

$$\int_\Omega \frac{f_k(x)}{\left(u_k(x)+\frac{1}{k}\right)^\gamma}\mathcal{G}_{t,h}(u_k(x))\,\mathrm{d}x \le h||f||_{L^\infty(\Omega)}\int_{u_k(x)>t+h}\frac{1}{\left(u_k(x)+\frac{1}{k}\right)^\gamma}\,\mathrm{d}x$$

$$+||f||_{L^\infty(\Omega)}\int_{t<u_k(x)\le t+h}\frac{u_k(x)-t}{\left(u_k(x)+\frac{1}{k}\right)^\gamma}\,\mathrm{d}x,$$

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we immediately get

$$\lim_{h\to 0^+}\frac{1}{h}\int_\Omega \frac{f_k(x)}{\left(u_k(x)+\frac{1}{k}\right)^\gamma}\mathcal{G}_{t,h}(u_k(x))\,\mathrm{d}x \le ||f||_{L^\infty(\Omega)}\int_{u_k(x)>t}\frac{1}{\left(u_k(x)+\frac{1}{k}\right)^\gamma}\,\mathrm{d}x$$

$$= N\omega_N||f||_{L^\infty(\Omega)}\int_0^{r(t)}\frac{1}{\left(\mathrm{u}_k(r)+\frac{1}{k}\right)^\gamma}r^{N-1}\,\mathrm{d}r.$$

We finally obtain that, for $0 \leq t < ||u_k||_{L^\infty(\Omega)}$, the following inequality holds true

$$\gamma(N,s)\int_0^{r(t)}\left(\int_{r(t)}^{+\infty}(\mathrm{u}_k(r)-\mathrm{u}_k(\rho))\Theta_{N,s}(r,\rho)\rho^{N-1}\,\mathrm{d}\rho\right)r^{N-1}\,\mathrm{d}r$$

$$\leq ||f||_{L^\infty(\Omega)}\int_0^{r(t)}\frac{1}{\left(\mathrm{u}_k(r)+\frac{1}{k}\right)^\gamma}r^{N-1}\,\mathrm{d}r.$$

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$$\leq ||f||_{L^\infty(\Omega)}\int_0^{r(t)}\frac{1}{\left(\mathrm{u}_k(r)+\frac{1}{k}\right)^\gamma}r^{N-1}\,\mathrm{d}r.$$

Reasoning as in Ferone and Volzone's paper, we can actually prove that, for every $r \geq 0$,

$$\gamma(N,s)\int_0^{r}\left(\int_{r}^{+\infty}(\mathrm{u}_k(\tau)-\mathrm{u}_k(\rho))\Theta_{N,s}(\tau,\rho)\rho^{N-1}\,\mathrm{d}\rho\right)\tau^{N-1}\,\mathrm{d}\tau$$

$$\leq ||f||_{L^\infty(\Omega)}\int_0^{r}\frac{1}{\left(\mathrm{u}_k(\tau)+\frac{1}{k}\right)^\gamma}\tau^{N-1}\,\mathrm{d}\tau.$$

Step 3. Symmetrized approximating problems

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Let v be the solution to the symmetrized problem (3). We denote by v_k the solution to the following problem

$$\begin{cases} (-\Delta)^s(v_k) = \dfrac{||f||_{L^\infty(\Omega)}}{\left(v_k + \frac{1}{k}\right)^\gamma} & \text{in } \Omega^\star \\ v_k > 0 & \text{in } \Omega^\star \\ v_k = 0 & \text{on } \mathbb{R}^N \setminus \Omega^\star. \end{cases} \tag{12}$$

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Due to the radial symmetry and the radial monotonicity, the function $\vartheta_k(x) = \vartheta_k(|x|) = v_k^\star(x)$ satisfies

$$\gamma(N,s) \int_0^r \left(\int_r^{+\infty} (\vartheta_k(\tau) - \vartheta_k(\rho)) \Theta_{N,s}(\tau,\rho) \rho^{N-1} \,\mathrm{d}\rho \right) \tau^{N-1} \,\mathrm{d}\tau$$

$$= ||f||_{L^\infty(\Omega)} \int_0^r \frac{1}{\left(\vartheta_k(\tau) + \frac{1}{k}\right)^\gamma} \tau^{N-1} \,\mathrm{d}\tau.$$

Step 4. Comparison result

Taking the difference between the inequality and the equality solved respectively by $\mathfrak{u}_k(x)$ and $\vartheta_k(x)$ we get

$$\int_0^r \left(\int_r^{+\infty} \left((\mathfrak{u}_k(\tau) - \vartheta_k(\tau)) - (\mathfrak{u}_k(\rho) - \vartheta_k(\rho)) \right) \Theta_{N,s}(\tau,\rho)\, \rho^{N-1}\, \mathrm{d}\rho \right) \tau^{N-1}\, \mathrm{d}\tau$$

$$\leq ||f||_{L^\infty(\Omega)} \int_0^r \left(\frac{1}{\left(\mathfrak{u}_k(\rho) + \frac{1}{k}\right)^\gamma} - \frac{1}{\left(\vartheta_k(\rho) + \frac{1}{k}\right)^\gamma} \right) \tau^{N-1}\, \mathrm{d}\tau.$$

We want to prove that

$$\int_0^r \mathfrak{u}_k(\tau)\tau^{N-1}\, \mathrm{d}\tau \leq \int_0^r \vartheta_k(\tau)\tau^{N-1}\, \mathrm{d}\tau, \qquad r \geq 0. \tag{13}$$

From now, our approach differs from the one used in the direct method used in Ferone and Volzone's paper, which consists in the interpretation of the LHS of the previous inequality as the difference of $N+2$ dimensional fractional Laplacian of the spherical mean functions of $\mathfrak{u}_k$, $\mathfrak{v}_k$. Indeed, we use now a qualitative contradiction argument based on Lemma "Max/Min".

Lemma (Max/Min)

Let u, v be two nonnegative, continuous functions on $[0, R]$. Let us define

$$H_u(r) = \int_0^r u(\rho)\rho^{N-1}\, \mathrm{d}\rho \qquad H_v(r) = \int_0^r v(\rho)\rho^{N-1}\, \mathrm{d}\rho \tag{14}$$

and

$$K_u(r) = \int_r^R u(\rho)\rho^{N-1}\, \mathrm{d}\rho \qquad K_v(r) = \int_r^R v(\rho)\rho^{N-1}\, \mathrm{d}\rho. \tag{15}$$

Assume that $H_u(r) - H_v(r)$ admits a positive maximum point at $\bar{r} > 0$, that is,

$$0 < H_u(\bar{r}) - H_v(\bar{r}) = \max_{r\in[0,R]} (H_u(r) - H_v(r)). \tag{16}$$

Then, if $h \not\equiv 0$ is a positive, increasing bounded function on $(0, R)$, we have

$$\int_0^{\bar{r}} u(\rho)h(\rho)\rho^{N-1}\, \mathrm{d}\rho - \int_0^{\bar{r}} v(\rho)h(\rho)\rho^{N-1}\, \mathrm{d}\rho > 0. \tag{17}$$

Analogously, assume that $K_u(r) - K_v(r)$ admits a negative minimum point at $\bar{r} < R$, that is,

$$0 > K_u(\bar{r}) - K_v(\bar{r}) = \min_{r \in [0,R]} (K_u(r) - K_v(r)).$$

Then, if $h \not\equiv 0$ is a positive, decreasing bounded function on $(0, R)$, we have

$$\int_{\bar{r}}^{R} u(\rho) h(\rho) \rho^{N-1} \, \mathrm{d}\rho - \int_{\bar{r}}^{R} v(\rho) h(\rho) \rho^{N-1} \, \mathrm{d}\rho < 0.$$

Suppose by contradiction that the function $\int_0^r (\mathfrak{u}_k(\tau) - \vartheta_k(\tau))\, \tau^{N-1}\, \mathrm{d}\tau$ has a positive maximum point at $\bar{r} \in (0, R]$, i.e.,

$$0 < \int_0^{\bar{r}} (\mathfrak{u}_k(\tau) - \vartheta_k(\tau))\, \tau^{N-1}\, \mathrm{d}\tau = \max_{r\in[0,R]} \int_0^r (\mathfrak{u}_k(\tau) - \vartheta_k(\tau))\, \tau^{N-1}\, \mathrm{d}\tau. \quad (18)$$

We recall that the function $\Theta_{N,s}(\tau, \rho)$ is increasing with respect to τ for any fixed $\rho > \bar{r}$.

Hence, Lemma Max/Min provides that, for every $\rho > \bar{r}$,

$$\int_0^{\bar{r}} (\mathrm{u}_k(\tau) - \vartheta_k(\tau))\Theta_{N,s}(\tau,\rho)\tau^{N-1}\,\mathrm{d}\tau > 0.$$

For what we have noticed before, if $\int_0^r (\mathrm{u}_k(\tau) - \vartheta_k(\tau))\,\tau^{N-1}\,\mathrm{d}\tau$ has a point of positive maximum at $\bar{r}$, then $\bar{r}$ is a point of non-positive minimum for $\int_r^R (\mathrm{u}_k(\tau) - \vartheta_k(\tau))\,\tau^{N-1}\,\mathrm{d}\tau$. Hence, using also the fact that $\Theta_{N,s}(\tau,\rho)$ is decreasing with respect to ρ for any fixed $\tau < \bar{r}$, we get that, for every $\tau < \bar{r}$,

$$\int_{\bar{r}}^R (\mathrm{u}_k(\rho) - \vartheta_k(\rho))\Theta_{N,s}(\tau,\rho)\rho^{N-1}\,\mathrm{d}\rho \le 0.$$

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$$\int_{\bar{r}}^{R} (\mathrm{u}_k(\rho) - \vartheta_k(\rho))\Theta_{N,s}(\tau,\rho)\rho^{N-1}\,\mathrm{d}\rho \leq 0.$$

We immediately deduce that

$$\int_0^{\bar{r}} \left(\int_{\bar{r}}^{+\infty} (\mathrm{u}_k(\tau) - \mathrm{u}_k(\rho))\Theta_{N,s}(\tau,\rho)\rho^{N-1}\,\mathrm{d}\rho\right)\tau^{N-1}\,\mathrm{d}\tau$$
$$-\int_0^{\bar{r}} \left(\int_{\bar{r}}^{+\infty} (\vartheta_k(\tau) - \vartheta_k(\rho))\Theta_{N,s}(\tau,\rho)\rho^{N-1}\,\mathrm{d}\rho\right)\tau^{N-1}\,\mathrm{d}\tau > 0.$$

Using the following Lemma

Lemma

Let $\gamma > 0$. Then, for every $a, b > 0$, we have

$$\frac{1}{a^\gamma} - \frac{1}{b^\gamma} \leq \frac{\gamma}{a^{\gamma+1}}(b-a). \tag{19}$$

with the choice $a = \mathfrak{u}_k(\tau) + \frac{1}{k}$ and $b = \vartheta_k(\tau) + \frac{1}{k}$, we get that

$$\int_0^{\bar{r}} \left(\frac{1}{\left(\mathfrak{u}_k(\tau) + \frac{1}{k}\right)^\gamma} - \frac{1}{\left(\vartheta_k(\tau) + \frac{1}{k}\right)^\gamma} \right) \tau^{N-1}\, \mathrm{d}\tau$$

$$\leq \gamma \int_0^{\bar{r}} \frac{1}{\left(\mathfrak{u}_k(\tau) + \frac{1}{k}\right)^{\gamma+1}} \left(\vartheta_k(\tau) - \mathfrak{u}_k(\tau)\right) \tau^{N-1}\, \mathrm{d}\tau,$$

the last integral being negative via Lemma of Max/Min since $\frac{1}{\left(u_k(\tau)+\frac{1}{k}\right)^{\gamma+1}}$ is a positive, increasing function.

This implies

$$\int_0^{\bar{r}} \left(\frac{1}{\left(\mu_k(\tau) + \frac{1}{k}\right)^{\gamma}} - \frac{1}{\left(\vartheta_k(\tau) + \frac{1}{k}\right)^{\gamma}} \right) \tau^{N-1} \, \mathrm{d}\tau < 0. \tag{20}$$

Finally,

$$\int_0^{\bar{r}} \left(\int_{\bar{r}}^{+\infty} (\mu_k(\tau) - \mu_k(\rho)) \Theta_{N,s}(\tau, \rho) \rho^{N-1} \, \mathrm{d}\rho \right) \tau^{N-1} \, \mathrm{d}\tau$$

$$- \int_0^{\bar{r}} \left(\int_{\bar{r}}^{+\infty} (\vartheta_k(\tau) - \vartheta_k(\rho)) \Theta_{N,s}(\tau, \rho) \rho^{N-1} \, \mathrm{d}\rho \right) \tau^{N-1} \, \mathrm{d}\tau > 0.$$

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This implies

$$\int_0^{\bar r}\left(\frac{1}{\left(\mathfrak{u}_k(\tau)+\frac{1}{k}\right)^{\gamma}}-\frac{1}{\left(\vartheta_k(\tau)+\frac{1}{k}\right)^{\gamma}}\right)\tau^{N-1}\,\mathrm{d}\tau<0. \tag{20}$$

Finally,

$$\int_0^{\bar r}\left(\int_{\bar r}^{+\infty}(\mathfrak{u}_k(\tau)-\mathfrak{u}_k(\rho))\Theta_{N,s}(\tau,\rho)\rho^{N-1}\,\mathrm{d}\rho\right)\tau^{N-1}\,\mathrm{d}\tau$$
$$-\int_0^{\bar r}\left(\int_{\bar r}^{+\infty}(\vartheta_k(\tau)-\vartheta_k(\rho))\Theta_{N,s}(\tau,\rho)\rho^{N-1}\,\mathrm{d}\rho\right)\tau^{N-1}\,\mathrm{d}\tau>0.$$

and

$$\int_0^{\bar r}\left(\frac{1}{\left(\mathfrak{u}_k(\tau)+\frac{1}{k}\right)^{\gamma}}-\frac{1}{\left(\vartheta_k(\tau)+\frac{1}{k}\right)^{\gamma}}\right)\tau^{N-1}\,\mathrm{d}\tau<0.$$

bring us to a contraddiction at $r=\bar r$.

5. Passing to the limit as $k \to +\infty$

In de Bonis, Barrios, Medina, Peral it has been proved that the sequences u_k, v_k are bounded in $X_0^s(\Omega)$, resp. $X_0^s(\Omega^\star)$. Hence, up to subsequences, u_k, v_k converge to functions $u \in X_0^s(\Omega)$, resp. $v \in X_0^s(\Omega^\star)$, weakly in X_0^s, strongly in L^p for any $p \in [1, 2_s^*)$ and a.e. in Ω, resp. $\Omega^\star$. Moreover, u, resp. v, are solutions to problems (1), resp. (3). Hence we can pass to the limit getting

$$\int_0^r \mathfrak{u}(\tau)\tau^{N-1}\, \mathrm{d}\tau \le \int_0^r \mathfrak{v}(\tau)\tau^{N-1}\, \mathrm{d}\tau, \qquad r \ge 0,$$

where $\mathfrak{u}(x) = \mathfrak{u}(|x|) = u^\star(x)$ and $\mathfrak{v}(x) = \mathfrak{v}(|x|) = v^\star(x)$.

B. Barrios, I. de Bonis, M. Medina, and I. Peral,, *Semilinear problems for the fractional laplacian with a singular nonlinearity,* Open Mathematics 13 (2015), 390–407.

Sketch of the proof of Theorem 2:

We consider the same sequence of approximating problems that we examined before and, for $0 \leq t < ||u_k||_{L^\infty(\Omega)}$ and $h > 0$, we choose $\varphi = u_k^\gamma \mathcal{G}_{t,h}(u_k^{\gamma+1})$ as test function in the weak formulation having

$$\frac{\gamma(N,s)}{2} \iint_{\mathbb{R}^{2N}} \frac{[u_k(x) - u_k(y)] \left[u_k(x)^\gamma \mathcal{G}_{t,h}\left(u_k(x)^{\gamma+1}\right) - u_k(y)^\gamma \mathcal{G}_{t,h}\left(u_k(y)^{\gamma+1}\right)\right]}{|x-y|^{N+2s}} \, \mathrm{d}x \, \mathrm{d} \tag{21}$$

$$= \int_\Omega \frac{f_k(x)}{\left(u_k(x) + \frac{1}{k}\right)^\gamma} u_k(x)^\gamma \mathcal{G}_{t,h}\left(u_k(x)^{\gamma+1}\right) \mathrm{d}x.$$

We use now the following Proposition:

Proposition

$\Phi : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous, convex function, such that $\Phi(0) = 0$. Then, if $u \in X_0^s(\Omega)$, we have

$$(-\Delta)^s \Phi(u) \le \Phi'(u)(-\Delta)^s u \qquad \text{weakly in } \Omega, \tag{22}$$

in the sense that for all nonnegative $\varphi \in X_0^s(\Omega)$ we have

$$\begin{aligned} &\iint_{\mathbb{R}^{2N}} \frac{[\Phi(u(x)) - \Phi(u(y))][\varphi(x) - \varphi(y)]}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \\ &\quad \le \iint_{\mathbb{R}^{2N}} \frac{[u(x) - u(y)][\Phi'(u(x))\,\varphi(x) - \Phi'(u(y))\,\varphi(y)]}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{23}$$

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T. Leonori, I. A. Peral, A. Primo, and F. Soria,, *Basic estimates for solutions of a class of nonlocal elliptic and parabolic equations,* Discrete and Continuous Dynamical Systems 35 (12) (2015), 6031–6068.

Using the inequality of the Proposition with the choices

$$\Phi(\theta) = \theta^{\gamma+1}, \quad \varphi(x) = \mathcal{G}_{t,h}\left(u_k(x)^{\gamma+1}\right)$$

we can estimate the left hand side of the previous inequality as follows

$$\iint_{\mathbb{R}^{2N}} \frac{[u_k(x) - u_k(y)]\left[u_k(x)^{\gamma}\mathcal{G}_{t,h}\left(u_k(x)^{\gamma+1}\right) - u_k(y)^{\gamma}\mathcal{G}_{t,h}\left(u_k(y)^{\gamma+1}\right)\right]}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y$$
$$\geq \frac{1}{\gamma+1}\iint_{\mathbb{R}^{2N}} \frac{\left[u_k(x)^{\gamma+1} - u_k(y)^{\gamma+1}\right]\left[\mathcal{G}_{t,h}\left(u_k(x)^{\gamma+1}\right) - \mathcal{G}_{t,h}\left(u_k(y)^{\gamma+1}\right)\right]}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y. \tag{24}$$

Reasoning as in the previous section, we can show that

$$\int\int_{\mathbb{R}^{2N}} \frac{\left(u_k(x)^{\gamma+1} - u_k(y)^{\gamma+1}\right)\left(\mathcal{G}_{t,h}(u_k(x)^{\gamma+1}) - \mathcal{G}_{t,h}(u_k(y)^{\gamma+1})\right)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y$$

$$\geq \int\int_{\mathbb{R}^{2N}} \frac{\left(u_k^\star(x)^{\gamma+1} - u_k^\star(y)^{\gamma+1}\right)\left(\mathcal{G}_{t,h}(u_k^\star(x)^{\gamma+1}) - \mathcal{G}_{t,h}(u_k^\star(y)^{\gamma+1})\right)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y$$

and

$$\lim_{h\to 0^+} \frac{1}{h}\int\int_{\mathbb{R}^{2N}} \frac{\left(u_k^\star(x)^{\gamma+1} - u_k^\star(y)^{\gamma+1}\right)\left(\mathcal{G}_{t,h}(u_k^\star(x)^{\gamma+1}) - \mathcal{G}_{t,h}(u_k^\star(y)^{\gamma+1})\right)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^{r(t)} \left(\int_{r(t)}^{+\infty} (\mathfrak{u}_k(r)^{\gamma+1} - \mathfrak{u}_k(\rho)^{\gamma+1})\Theta_{N,s}(r,\rho)\rho^{N-1}\,\mathrm{d}\rho\right) r^{N-1}\,\mathrm{d}r.$$

where, as in the previous proof, $\mathfrak{u}_k(x) = \mathfrak{u}_k(|x|)$ stands for $u_k^\star(x)$.

Regarding the lower order term in the right hand side, we observe that

$$\int_\Omega \frac{f_k(x)}{\left(u_k(x)+\frac{1}{k}\right)^\gamma}\, u_k(x)^\gamma \mathcal{G}_{t,h}(u_k(x)^{\gamma+1})\,\mathrm{d}x = h\int_{u_k^{\gamma+1}>t+h} \frac{f_k(x)}{\left(u_k(x)+\frac{1}{k}\right)^\gamma}\, u_k(x)^\gamma\,\mathrm{d}x$$

$$+\int_{t<u_k^{\gamma+1}\le t+h} \frac{f_k(x)}{\left(u_k(x)+\frac{1}{k}\right)^\gamma}\, u_k(x)^\gamma (u_k(x)^{\gamma+1}-t)\,\mathrm{d}x$$

and

$$\int_{t<u_k^{\gamma+1}\le t+h} \frac{f_k(x)}{\left(u_k(x)+\frac{1}{k}\right)^\gamma}\, u_k(x)^\gamma (u_k(x)^{\gamma+1}-t)\,\mathrm{d}x \le h\,||f||_\infty \int_{t<u_k^{\gamma+1}\le t+h}\mathrm{d}x.$$

It follows that

$$\lim_{h\to 0^+}\frac{1}{h}\int_\Omega \frac{f_k(x)}{\left(u_k(x)+\frac{1}{k}\right)^\gamma}\, u_k(x)^\gamma \mathcal{G}_{t,h} u_k(x)^{\gamma+1}\,\mathrm{d}x =$$

$$\int_{u_k^{\gamma+1}>t} \frac{f_k(x)}{\left(u_k(x)+\frac{1}{k}\right)^\gamma}\, u_k(x)^\gamma\,\mathrm{d}x.$$

It is easy to observe that

$$\int_{u_k^{\gamma+1}>t} \frac{f_k(x)}{\left(u_k(x)+\frac{1}{k}\right)^{\gamma}}\, u_k(x)^{\gamma}\ \mathrm{d}x \leq \int_{u_k^{\gamma+1}>t} f(x)\ \mathrm{d}x \leq \int_0^{r(t)} f^{\star}(\rho)\rho^{N-1}\ \mathrm{d}\rho,$$

being $\mathfrak{u}_k(r(t))^{\gamma+1} = t$.

We deduce

$$\frac{\gamma(N,s)}{2(\gamma+1)}\int_0^{r(t)}\left(\int_{r(t)}^{+\infty}(\mathrm{u}_k(r)^{\gamma+1}-\mathrm{u}_k(\rho)^{\gamma+1})\Theta_{N,s}(r,\rho)\rho^{N-1}\,\mathrm{d}\rho\right)r^{N-1}\,\mathrm{d}r$$

$$\leq \int_0^{r(t)} f^\star(\rho)\rho^{N-1}\,\mathrm{d}\rho.$$

Reasoning as Ferone, Volzone's paper, we can show that actually the following inequality holds true for every $r\geq 0$:

$$\frac{\gamma(N,s)}{2(\gamma+1)}\int_0^{r}\left(\int_{r}^{+\infty}(\mathrm{u}_k(\tau)^{\gamma+1}-\mathrm{u}_k(\rho)^{\gamma+1})\Theta_{N,s}(\tau,\rho)\rho^{N-1}\,\mathrm{d}\rho\right)\tau^{N-1}\,\mathrm{d}\tau$$

$$\leq \int_0^{r} f^\star(\rho)\rho^{N-1}\,\mathrm{d}\rho.$$

On the other hand, the solution to problem solved by v satisfies

$$\frac{\gamma(N,s)}{2(\gamma+1)}\int_0^r\left(\int_r^{+\infty}(\vartheta_k(\tau)-\vartheta_k(\rho))\Theta_{N,s}(\tau,\rho)\rho^{N-1}\,\mathrm{d}\rho\right)\tau^{N-1}\,\mathrm{d}\tau$$

$$=\int_0^r f^\star(\rho)\rho^{N-1}\,\mathrm{d}\rho.$$

Taking the difference between the inequality and the equality solved respectively by $\mathfrak{u}_k(x)$ and $\vartheta_k(x)$ we can conclude as in the proof of Theorem 1.

Regularity results

As an immediate consequence of Theorem 2 we can prove the following regularity results, depending on the value of γ and on the summability of f.

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Theorem

Let $s \in (0,1)$, $N \geq 2$, $\gamma > 0$, and assume that $f \in L^p(\Omega)$, with $p \geq 2^*_s$, $f \geq 0$. If $u \in X^s_0(\Omega)$ is the weak solution to problem (1), the following estimates hold true.

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- *If $p < \frac{N}{2s}$, then $u \in L^q(\Omega)$, with $q = \frac{Np(\gamma+1)}{N-2sp}$, and there exists a positive constant C such that*

$$||u||_{L^q(\Omega)} \leq C||f||_{L^p(\Omega)}^{1/(\gamma+1)}.$$

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$$||u||_{L^q(\Omega)} \leq C||f||_{L^p(\Omega)}^{1/(\gamma+1)}.$$
- *If $p = \frac{N}{2s}$, then $u \in L_\Phi(\Omega)$, where $L_\Phi(\Omega)$ is the Orlicz space generated by the N-function*
$$\Phi(t) = \exp(|t|^{(\gamma+1)p'}) - 1.$$
Moreover, there exists a positive constant C such that
$$||u||_{L_\Phi(\Omega)} \leq C||f||_{L^p(\Omega)}^{1/(\gamma+1)}.$$

- If $p > \frac{N}{2s}$, then $u \in L^\infty(\Omega)$ and there exists a positive constant C such that

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Remark: We stress that when $\gamma = 0$ we recover the estimates contained in Ferone and Volzone's paper, while when $s = 1$ we have the same estimates contained in Boccardo and Orsina's paper and Brandolini, Chiacchio and Trombetti's paper.

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V. Ferone, and B. Volzone, *Symmetrization for fractional elliptic problems: a direct approach*, Arch. Rational Mech. Anal. 239 (2021), 1733–1770.

THANKS FOR YOUR ATTENTION!!!